

INVARIANT THEORY OF $\bigwedge^3(9)$ AND GENUS 2 CURVES

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ABSTRACT. Previous work established a connection between the geometric invariant theory of the third exterior power of a 9-dimensional complex vector space and the moduli space of genus 2 curves with some additional data. We generalize this connection to arbitrary fields, and describe the arithmetic data needed to get a bijection between both sides of this story.

1. INTRODUCTION

This paper is a companion to our previous paper [RS]. We begin by briefly recalling what was done there. Given a genus 2 curve C over a field \mathbf{k} , let $\mathrm{SU}_3(C)$ be the coarse moduli space of rank 3 semistable vector bundles on C . It admits a degree 2 map $\mathrm{SU}_3(C) \rightarrow \mathbf{P}^8$ which is branched along a sextic hypersurface. Remarkably, the singular locus of the projective dual of this sextic is a surface which is isomorphic to the Jacobian of C over the algebraic closure of \mathbf{k} . This story has been developed over algebraically closed fields of characteristic 0 in [O, Min] and connected to the invariant theory of the action of $\mathbf{SL}_9(\mathbf{k})$ on $\bigwedge^3 \mathbf{k}^9$ in [GS2, GSW]. In [RS], the setting is generalized to arbitrary fields, and the purpose of this paper is to extend the invariant-theoretic aspects.

More precisely, let V be a 9-dimensional vector space over an arbitrary field \mathbf{k} and consider the action of $\mathbf{SL}(V)$ on $\bigwedge^3 V$. Given a stable (in the sense of geometric invariant theory) element $\gamma \in \bigwedge^3 V$, we generalize the constructions in [GSW, GS2] to produce:

- a genus 2 curve C with a Weierstrass point $P \in C(\mathbf{k})$,
- a cubic hypersurface in $\mathbf{P}(V^*)$ whose singular locus is a smooth surface X , and
- a sextic hypersurface in $\mathbf{P}(V)$,

such that:

- the cubic and sextic hypersurfaces are projectively dual to one another, and
- X is isomorphic to the Jacobian $J(C)$ of C over the algebraic closure of \mathbf{k} .

In fact, we also get some interesting arithmetic data:

- a 3-covering $X \rightarrow J(C)$ which becomes the multiplication by 3 map over $\bar{\mathbf{k}}$, i.e., an element in $H^1(\mathbf{k}; J(C)[3])$; furthermore, it lies in the kernel of a map $H^1(\mathbf{k}; J(C)[3]) \rightarrow H^1(\mathbf{k}; \mathbf{SL}_9/\mu_3)$.

Conversely, given this data, we show how to construct a stable element in $\bigwedge^3 V$ (which is only well-defined up to scalar multiple and the action of $\mathbf{SL}(V)$). A bulk of the work in this paper is to show that these two constructions are inverse to one another.

Our work is partially motivated by recent work in “arithmetic invariant theory” (see [BG] for example). One goal is to count arithmetic objects of interest, and the first step in many of these cases is to parametrize them by orbits in a linear space. This first step is achieved

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here; when \mathbf{k} is a global field, we show that the 3-Selmer group of $J(C)$ is a subgroup of the kernel of $H^1(\mathbf{k}; J(C)[3]) \rightarrow H^1(\mathbf{k}; \mathbf{SL}_9/\mu_3)$, so that, in fact, they are parametrized by special kinds of orbits in $\bigwedge^3 V$. Following the analogies of previous work in the area, we may hope to count the average size of this 3-Selmer group using $\bigwedge^3 V$.

Here is a brief overview of the contents. In §2, we work out the aspects of the invariant theory of $\bigwedge^3 V$ which are needed in the rest of the paper. In §3, we generalize the construction of [GS2, GSW] to arbitrary fields, i.e., produce the data above starting from a stable element γ . In §4, we provide a construction in the reverse direction: starting from the data above, we produce a stable element γ , and in §5, we show that these two constructions are inverse to one another. Finally, in §6, we discuss a few additional topics: Selmer groups, ordinary curves, and an explicit model for the 3-torsion of $J(C)$ given γ above.

1.1. Notation. \mathbf{k} is a field and R is a complete discrete valuation ring (DVR from now on) of characteristic 0 whose residue field is \mathbf{k} ; the quotient field of R is denoted \mathbf{K} . Write \mathbf{k}^{sep} for a separable closure of \mathbf{k} .

If G is a group scheme defined over \mathbf{k} , we let $H^*(\mathbf{k}; G)$ denote the flat cohomology of G . When G is smooth, this coincides with the Galois cohomology of G , but we will not have any use for Galois cohomology of non-smooth group schemes.

2. INVARIANT THEORY PRELIMINARIES

2.1. Geometric invariant theory review. Let G be a reductive group acting linearly on a vector space V . A point $u \in V$ is **stable** if its stabilizer subgroup in G is finite and its orbit is closed, and it is **semistable** if 0 is not in the closure of its orbit. If u is not semistable, then it is **unstable**. Hence, an element is “non-stable” if it is unstable or if it is semistable, but not stable.

The Hilbert–Mumford criterion says that u is stable if and only if $\lim_{t \rightarrow 0} \rho(t).u$ does not exist for any 1-parameter subgroup $\rho: \mathbf{G}_m \rightarrow G$, and that u is semistable if and only if $\lim_{t \rightarrow 0} \rho(t).u$ does not exist, or it is nonzero whenever the limit exists.

The set of unstable points form a Zariski closed set, and is the zero locus of all positive degree G -invariant homogeneous polynomials on V . Similarly, the set of non-stable points form a Zariski closed set. Finally, two points $x, y \in V$ are **S-equivalent** if $f(x) = f(y)$ for all homogeneous G -invariant polynomials f on V , and they are **projectively S-equivalent** if αx is S-equivalent to y for some $\alpha \neq 0$. If x and y are S-equivalent semistable points, then their orbit closures have a semistable point in common. Furthermore, the orbit closure of any semistable point x contains a unique closed orbit of semistable points, and if x is not stable, then neither are the points of this closed orbit.

Let V_9 denote a vector space of dimension 9 with basis e_1, \dots, e_9 . The group $G = \mathbf{SL}(V_9)/\mu_3$ acts on $\bigwedge^3(V_9)$, and the invariant theory of this representation is the main focus of this paper (see also [EV] for earlier work). It has a natural basis of monomials $e_i \wedge e_j \wedge e_k$ (with $1 \leq i < j < k \leq 9$), and we will use $[ijk]$ as shorthand for this monomial.

For the following, see [GGR].

Proposition 2.1. *Over an algebraically closed field, every element of $\bigwedge^3(V_9)$ is S-equivalent to an element of the form*

$$\begin{aligned} & [267] + [258] + [348] + [169] + [357] + [249] + [178] + [456] \\ & - c_3[257] - c_6[247] + c_9[148] - c_{12}[147] + c_{15}[235] + c_{18}[145] + c_{24}[134] + c_{30}[123], \end{aligned}$$

where if 2 is invertible we may take $c_3 = c_9 = c_{15} = 0$ and if 5 is invertible we may take $c_6 = 0$. Two such elements are projectively S -equivalent if and only if the corresponding curves

$$C_c : x^2 + z^5 + c_3xz^2 + c_6z^4 + c_9xz + c_{12}z^3 + c_{15}x + c_{18}z^2 + c_{24}z + c_{30} = 0$$

are isomorphic.

Remark 2.2. Above, we see that projective S -equivalence classes classify pairs (C, P) where C is a genus 2 curve and $P \in C(\mathbf{k})$ is a rational Weierstrass point. In fact, one can show that the S -equivalence classes themselves classify triples (C, P, φ) where $\varphi: \omega_C \otimes \mathcal{O}_P \cong \mathcal{O}_P$ specifies a nonzero tangent vector at P . We omit the details, as we have not been able to figure out how to build φ into the construction below, and can thus only work at the level of projective S -equivalence. \square

2.2. Cartan subspaces. Assume the characteristic of \mathbf{k} is different from 3. Let $G = \mathbf{SL}(V_9)/\mu_3$ and let \mathfrak{e}_8 be the split Lie algebra of type E_8 and let Γ be its simply-connected group. We have a $\mathbf{Z}/3$ -graded decomposition

$$(2.3) \quad \mathfrak{e}_8 = \mathfrak{sl}(V_9) \oplus \bigwedge^3 V_9 \oplus \bigwedge^6 V_9.$$

The decomposition (2.3) corresponds to an order 3 automorphism θ of Γ such that $G = \Gamma^\theta$ and $\bigwedge^3 V_9$ is one of the nontrivial eigenspaces of θ acting on \mathfrak{e}_8 .

The 4 dimensional subspace \mathfrak{h} of $\bigwedge^3 V_9$ spanned by

$$(2.4) \quad \begin{aligned} &[123] + [456] + [789], & [147] + [258] + [369], \\ &[159] + [267] + [348], & [168] + [249] + [357], \end{aligned}$$

is the **standard Cartan subspace**. It may be helpful to visualize this in terms of the finite geometry $\mathbf{P}_{\mathbf{F}_3}^2$, namely, each basis vector is a sum over all lines in a direction of the following table:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}$$

This carries the action of the Weyl group $W = N(\mathfrak{h})/Z(\mathfrak{h})$ (normalizer modulo centralizer).

Proposition 2.5. *If \mathbf{k} has characteristic 0, the restriction map*

$$\mathbf{k}[\bigwedge^3 V_9]^G \xrightarrow{\cong} \mathbf{k}[\mathfrak{h}]^W$$

is an isomorphism, and both are polynomial rings generated by elements of degrees 12, 18, 24, 30.

Proof. See [V, Theorem 7] for the isomorphism, and see [V, §9] for the degrees of the invariants. \square

W is a complex reflection group (the reflections have order 3), and there are 40 reflection hyperplanes. With respect to the 4 basis vectors in (2.4) for the standard Cartan subspace, the matrix representation of the reflection group in characteristic 0 is given in [GS2, §3.1]. Each reflection hyperplane is in the orbit of the hyperplane spanned by the first 3 basis vectors (see [GS2, Table 1]).

Lemma 2.6. *Suppose \mathbf{k} has characteristic 0. If x is semistable and Gx is closed, then $Gx \cap \mathfrak{h} \neq 0$.*

Proof. Combine [V, Proposition 4] and [V, Corollary of Theorem 1]. \square

Proposition 2.7. *Any element of a reflection hyperplane in the standard Cartan subspace has a positive-dimensional stabilizer subgroup in G .*

Proof. In positive characteristic, lift our element over the DVR R to characteristic 0 and use semicontinuity of stabilizer dimension to reduce the proof to the case of characteristic 0.

The reflection hyperplanes form a single orbit under the reflection group, so it suffices to consider a single one. From the discussion above, we may assume that this hyperplane is the span of $[123] + [456] + [789]$, $[147] + [258] + [369]$, $[159] + [267] + [348]$. Then for any t , the diagonal matrix with entries $(t^{-2}, t, t, t, t, t^{-2}, t, t^{-2}, t)$ stabilizes each of these 3 basis vectors, and hence any element in this hyperplane. So the stabilizer of any element has positive dimension. \square

Proposition 2.8. *An element u in the standard Cartan subspace is stable if and only if it does not lie in any reflection hyperplanes.*

Proof. The standard Cartan subspace is the intersection of a Cartan subalgebra of \mathfrak{e}_8 with $\bigwedge^3 V_9$ and none of the reflection hyperplanes of the Cartan subalgebra of \mathfrak{e}_8 contain the standard Cartan subspace (this follows from the discussion in [El, §3]), so u is contained in the complement of reflection hyperplanes in a Cartan subalgebra of \mathfrak{e}_8 , which means that it is stable under the action of Γ . The Hilbert–Mumford criterion implies that u is stable as an element of $\bigwedge^3 V_9$ under the action of G . Conversely, we have already seen that any element in a reflection hyperplane has a positive-dimensional stabilizer, so cannot be stable. \square

2.3. Stable elements.

Lemma 2.9. *In characteristic 0, the locus of non-stable elements of $\bigwedge^3 V_9$ is contained in an irreducible G -invariant hypersurface of degree 120.*

Proof. Let x be a semistable, but not stable point, and let y be a point in its orbit closure such that Gy is closed. Then $Gy \cap \mathfrak{h} \neq 0$ by Lemma 2.6, and we may assume $y \in \mathfrak{h}$. By Proposition 2.8, y lies on a reflection hyperplane. Let f be the product of the linear forms vanishing on the reflection hyperplanes of \mathfrak{h} , so $\deg f = 40$. The reflections transform f by a cube root of unity, so f^3 is the lowest degree W -invariant vanishing on each reflection hyperplane. Let δ be the G -invariant function on $\bigwedge^3 V_9$ which corresponds to f^3 under the isomorphism in Proposition 2.5. Then δ vanishes on y since it restricts to f^3 , and hence δ also vanishes on x . Finally, δ is irreducible: if not, then each component is cut out by a G -invariant since G is connected, and would restrict to W -invariant function vanishing on some of the reflection hyperplanes of degree < 120 , but no such function exists. \square

Proposition 2.10. (a) *An element $u \in \bigwedge^3 V_9$ is non-stable if and only if there exists a 6-dimensional subspace $U \subset V_9$ such that $\gamma \in \bigwedge^3 U + \bigwedge^2 U \otimes (V_9/U)$.*
 (b) *The set of non-stable elements in $\bigwedge^3 V_9$ is an irreducible hypersurface which is set-theoretically defined by a polynomial of degree 120. This hypersurface is reduced in characteristic 0.*

Proof. Let Z be the set of u such that there exists a 6-dimensional subspace $U \subset V_9$ such that $\gamma \in \bigwedge^3 U + \bigwedge^2 U \otimes (V_9/U)$.

Pick $\gamma \in Z$ with U as above. Pick a basis u_1, \dots, u_6 for U and extend it to a basis u_1, \dots, u_9 for V_9 . Then γ is a sum of trivectors $[ijk]$ where $|\{i, j, k\} \cap \{7, 8, 9\}| \leq 1$. In particular,

given the diagonal 1-parameter subgroup $\rho(t) = (t^3, t^3, t^3, t^3, t^3, t^3, t^{-6}, t^{-6}, t^{-6})$, we have $\lim_{t \rightarrow 0} \rho(t) \cdot \gamma$ exists, and is the result of throwing away the $[ijk]$ where $|\{i, j, k\} \cap \{7, 8, 9\}| = 0$. By the Hilbert–Mumford criterion, γ is non-stable, so Z is contained in the non-stable locus.

Let P be the stabilizer in $\mathbf{GL}(V_9)$ of the subspace e_1, \dots, e_6 and let E be the span of e_1, \dots, e_6 . Then the span of $\Lambda^3 E$ and $\Lambda^2 E \otimes (V_9/E)$ is a P -submodule of $\Lambda^3 V_9$ and by algebraic induction, this P -submodule becomes a rank 65 vector bundle \mathcal{E} which is a subbundle of $\Lambda^3 V_9 \times \mathbf{Gr}(6, V_9)$ where $\mathbf{Gr}(6, V_9)$ is the Grassmannian of 6-dimensional subspaces of V_9 . By the discussion above, the image of the projection $\pi: \mathcal{E} \rightarrow \Lambda^3 V_9$ is Z . In particular, Z is irreducible. Let $\xi \subset \Lambda^3 V_9^* \times \mathbf{Gr}(6, V_9)$ be the annihilator of \mathcal{E} ; then the Koszul complex $\Lambda^\bullet \xi$ is a locally free resolution of \mathcal{E} as a subscheme of $\Lambda^3 V_9 \times \mathbf{Gr}(6, V_9)$, and so its derived pushforward with respect to π has the same Euler characteristic as $R\pi_* \mathcal{O}_{\mathcal{E}}$ (for a discussion of this, see [W, Chapter 5]). More specifically, everything respects the natural \mathbf{Z} -grading, so we can calculate the Hilbert series of $R\pi_* \mathcal{O}_{\mathcal{E}}$ as:

$$\sum_{i \geq 0} (-1)^i H_{R\pi_* \mathcal{O}_{\mathcal{E}}}(t) = \sum_{i=0}^{19} (-1)^i \chi(\mathbf{Gr}(6, V_9); \bigwedge^i \xi) \frac{t^i}{(1-t)^{84}}.$$

The right hand side can be computed using Borel–Weil–Bott [W, Corollary 4.1.7] and yields

$$\frac{1 + t^6 + t^9 + 81t^{18} - 84t^{19}}{(1-t)^{84}} = \frac{h(t)}{(1-t)^{83}}$$

where

$$\begin{aligned} h(t) = & 84t^{18} + 3t^{17} + 3t^{16} + 3t^{15} + 3t^{14} + 3t^{13} + 3t^{12} + 3t^{11} \\ & + 3t^{10} + 3t^9 + 2t^8 + 2t^7 + 2t^6 + t^5 + t^4 + t^3 + t^2 + t + 1. \end{aligned}$$

In particular, the support of $R\pi_* \mathcal{O}_{\mathcal{E}}$ has dimension 83, and this support is Z . This matches the dimension of the total space of \mathcal{E} , so generically, the map π has 0-dimensional fibers. Since π is projective, this implies that the support of $R^i \pi_* \mathcal{O}_{\mathcal{E}}$ for each $i > 0$ has dimension ≤ 82 . In particular, the multiplicity of $\pi_* \mathcal{O}_{\mathcal{E}}$ is $h(1) = 120$, and the degree of Z divides 120.

In characteristic 0, we know that Z is contained in an irreducible hypersurface of degree 120 by Lemma 2.9, so we conclude that Z coincides with this hypersurface. This proves (a) and (b) in characteristic 0.

Now we prove (a) in general. What remains is to show that every non-stable element belongs to Z . Let γ be a non-stable element. Let R be a complete DVR with residue field \mathbf{k} and fraction field \mathbf{K} of characteristic 0. Let ρ be a 1-parameter subgroup of $G(\mathbf{k})$ such that $\lim_{t \rightarrow 0} \rho(t) \cdot \gamma$ exists. By changing basis, we may assume that the image of ρ is contained in the diagonal matrices, and hence ρ can be lifted to a 1-parameter subgroup $\tilde{\rho}$ of $G(R)$. The action of $\tilde{\rho}$ on $\Lambda^3 R^9$ decomposes it into weight spaces which are free R -submodules, we are interested in the negative versus non-negative subspaces. The non-negative subspace corresponds to all elements which have a limit under the action of $\tilde{\rho}(t)$ for $t \rightarrow 0$ and its reduction to \mathbf{k} is the non-negative subspace of the action of ρ on $\Lambda^3 V_9$. So we can lift γ to a non-stable element $\tilde{\gamma} \in \Lambda^3 R^9$ such that $\tilde{\gamma}_{\mathbf{K}} \in \Lambda^3 \mathbf{K}^9$ is also non-stable. By what we just showed, there exists a 6-dimensional subspace $U \subset \mathbf{K}^9$ such that $\tilde{\gamma}_{\mathbf{K}} \in \Lambda^3 U + \Lambda^2 U \otimes (\mathbf{K}^9/U)$. Since the Grassmannian is proper, U can be lifted to a rank 6 R -submodule $\tilde{U} \subset R^9$ such that R^9/\tilde{U} is free. In particular, $\tilde{\gamma} \in \Lambda^3 \tilde{U} + \Lambda^2 \tilde{U} \otimes R^9/\tilde{U}$ since this is a closed condition on

the fibers of R and it is true generically. In particular, the special fiber of \tilde{U} gives a subspace which shows that $\gamma \in Z$.

By what was shown already, we know that Z is an irreducible hypersurface whose degree divides 120, so we conclude that the same is true for the non-stable locus. \square

Proposition 2.11. *γ_c is stable if and only if C_c is smooth.*

Proof. If the curve C_c is singular, then translating the singular point to $(0, 0)$ gives a curve

$$C_{c'} : x^2 + z^5 + c'_3 x z^2 + c'_6 z^4 + c'_9 x z + c'_{12} z^3 + c'_{15} x + c'_{18} z^2 + c'_{24} z + c'_{30} = 0.$$

In particular, $c'_{30} = 0$ (since $(0, 0)$ is a point) and $c'_{15} = c'_{24} = 0$ (since the partial derivatives of x and z vanish at $(0, 0)$). If we take $U = \langle e_4, e_5, e_6, e_7, e_8, e_9 \rangle$, then $\gamma_{c'} \in \bigwedge^3 U + \bigwedge^2 U \otimes (V_9/U)$, so $\gamma_{c'}$ is non-stable by Proposition 2.10, so the same is true for γ_c since they are projectively S-equivalent by Proposition 2.1.

In characteristics different from 2 and 5, the curve can be put in the form

$$x^2 + z^5 + c_{12} z^3 + c_{18} z^2 + c_{24} z + c_{30} = 0,$$

which is singular if and only if the quintic (in z) has discriminant 0. The discriminant of the quintic is the restriction to the space of γ_c 's of an invariant of degree 120, and we conclude that an S-equivalence class meets that space in a non-stable element when that invariant vanishes. In other words, the discriminant cuts out (a component of) the non-stable locus over $\mathbf{Z}[\frac{1}{10}]$. In particular, the corresponding hypersurface is irreducible, so the only way the hypersurface could fail to be integral is if the discriminant were a constant multiple of a power. Since it has a term $3125c_{30}^4$, if it were a power, it would necessarily be a square, and since it has a term $256c_{24}^5$, it cannot be a square.

Since the non-stable locus is a hypersurface in every characteristic by Proposition 2.10, we can obtain that hypersurface by clearing the denominator from the discriminant over $\mathbf{Z}[\frac{1}{10}]$. That is, if we take the general curve C_c , complete the square and fifth power, and then take the discriminant, the result will be a polynomial over $\mathbf{Z}[\frac{1}{10}]$, and multiplying by suitable powers of 2 and 5 gives a polynomial over \mathbf{Z} with nontrivial reductions modulo 2 and 5 which vanishes precisely when C_c is singular. These reductions are easily verified to be irreducible, and thus the non-stable locus is an irreducible hypersurface of degree 120 in all characteristics. \square

3. PARAMETRIZING 3-COVERINGS OF ABELIAN SURFACES

Let $(\bigwedge^3 V_9)_{\text{st}}$ be the set of stable elements of $\bigwedge^3 V_9$ with respect to the $\mathbf{SL}(V_9)/\mu_3$ -action. Fix $u \in (\bigwedge^3 V_9)_{\text{st}}$. From this data, we will construct:

- a genus 2 curve C with a marked Weierstrass point $P \in C(\mathbf{k})$,
- a 3-covering $\psi: X \rightarrow J$ (where $J = J(C)$ is the Jacobian of C) such that $[\psi] \in \ker(H^1(\mathbf{k}; J(C)[3]) \rightarrow H^1(\mathbf{k}; \mathbf{SL}(V_9)/\mu_3))$.

Recall that $\psi: X \rightarrow J$ is a 3-covering if X is a torsor for J and ψ can be identified with the multiplication-by-3 map over an algebraic closure of \mathbf{k} ; 3-coverings are classified by cohomology classes in $H^1(\mathbf{k}; J[3])$ [Sk, Proposition 3.3.2].

To simplify notation, we will not label the objects by u , but we emphasize that all constructions depend on the $\mathbf{PGL}(V_9)$ -orbit of $[u] \in \mathbf{P}((\bigwedge^3 V_9)_{\text{st}})$.

Let $\mathbf{P}(V_9^*)$ denote the space of lines in V_9^* . Then V_9 is the space of linear functions on $\mathbf{P}(V_9^*)$, so we can treat e_1, \dots, e_9 as coordinate functions. Following [GS2, §3.2], we interpret $u \in \Lambda^3 V_9$ as a family of 9×9 skew-symmetric matrices

$$\Phi: V_9^* \rightarrow V_9 \otimes \mathcal{O}_{\mathbf{P}(V_9^*)}(1)$$

over $\mathbf{P}(V_9^*)$. In more details, given $u \in \Lambda^3 V_9$, apply the comultiplication map $\Lambda^3 V_9 \rightarrow \Lambda^2 V_9 \otimes V_9$, use the natural surjection $V_9 \otimes \mathcal{O}_{\mathbf{P}(V_9^*)} \rightarrow \mathcal{O}_{\mathbf{P}(V_9^*)}(1)$, and interpret $\Lambda^2 V_9$ as the space of skew-symmetric matrices $V_9^* \rightarrow V_9$. In particular, this construction is $\mathbf{GL}(V_9)$ -equivariant, so acting by $\mathbf{GL}(V_9)$ amounts to a projective linear change of coordinates in $\mathbf{P}(V_9^*)$.

Let $Y \subset \mathbf{P}(V_9^*)$ be the locus where $\text{rank } \Phi \leq 6$. Let $X \subset \mathbf{P}(V_9^*)$ be the locus where $\text{rank } \Phi \leq 4$.

Lemma 3.1. *X is smooth of dimension 2, and the locus where $\text{rank } \Phi \leq 2$ is empty.*

Proof. If there is a point in the rank 2 locus, then we can choose a basis so that it is the point $[1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0]$. So $u = [123] + u'$ where no monomial in u' contains e_1 . Then $(e_1 \mapsto e_1 + \alpha e_2 + \beta e_3)$ is a 2-dimensional subgroup of the stabilizer of u , which contradicts that u has a finite stabilizer group. A similar argument works if there is a point in the rank 0 locus.

Let $\mathcal{V}_1 = \mathcal{O}(-1)|_X$ be the restriction of the tautological subbundle of lines to X . Also let $\mathcal{V}_9 = V_9^*|_X$ and $\mathcal{V}_5 = \ker \Phi|_X$. Then \mathcal{V}_5 is a rank 5 vector bundle on X satisfying $\mathcal{V}_1 \subset \mathcal{V}_5 \subset \mathcal{V}_9$.

We now compute the tangent space of $x \in X$. Do a change of basis so that $e_i(x) = 0$ for $i > 1$ and so that $(\mathcal{V}_5)_x$ is defined by $e_i = 0$ for $i > 5$. Then at the local ring of x (with maximal ideal \mathfrak{m}), possibly up to another change of basis, Φ is the direct sum of a 5×5 skew-symmetric matrix Φ' with entries in \mathfrak{m} with an invertible 4×4 skew-symmetric matrix. The derivatives of the 6×6 Pfaffians are 4×4 Pfaffians, so only 6×6 Pfaffians which use the last 4 rows/columns contribute to nonzero rows of the Jacobian map. Furthermore, these entries are identified with the entries of Φ' . So the tangent space of $x \in X$ is the kernel of the Jacobian map $\mathcal{V}_9/\mathcal{V}_1 \rightarrow \Lambda^2(\mathcal{V}_5/\mathcal{V}_1)$ restricted to x .

Hence it suffices to prove that the Jacobian map is surjective at all points of X . Suppose there is a point $x \in X$ so that the map is not surjective. Choose a nonzero linear functional λ that annihilates the image. The calculation is equivariant under $\mathbf{SL}((\mathcal{V}_5/\mathcal{V}_1)_x)$, so we only need to check what happens for a single representative in each orbit in $\Lambda^2(\mathcal{V}_5/\mathcal{V}_1)^*$.

If λ has rank 2 (say $\lambda(m)$ is the coefficient of $e_2 \wedge e_3$), it induces a subspace V_3 of $(\mathcal{V}_5)_x$ containing $(\mathcal{V}_1)_x$, and the 1-parameter subgroup with weight $(-2, -2, -2, 1, 1, 1, 1, 1, 1)$ is destabilizing. Indeed, before imposing the condition that λ annihilates the map to $\Lambda^2(V_5/V_1)$, the only monomials preventing that weight from destabilizing are $[23i]$ for $4 \leq i \leq 9$; let α_i be the coefficient of $[23i]$. So the $e_2 \wedge e_3$ entry, which is 0, is $\sum_{i=4}^9 \pm \alpha_i e_i$, so $\alpha_i = 0$ for all i .

If λ has rank 4, we similarly find that the weight $(-4, -1, -1, -1, -1, 2, 2, 2, 2)$ is destabilizing. Here the only monomials preventing that weight from destabilizing are $[ijk]$ where $\{i, j, k\} \subset \{2, 3, 4, 5\}$, and it is easy to see that they cannot appear. \square

Lemma 3.2. *Y is a cubic hypersurface whose singular locus is X .*

Proof. The fact that Y is a cubic hypersurface follows from [GSW, §5].

It follows from the chain rule that all partial derivatives of the cubic defining Y vanish on X , so we just need to show that Y is smooth away from X . Let \mathcal{V}_1 denote the restriction

of the tautological subbundle of lines to $Y \setminus X$. Note that $\mathcal{V}_1 = \mathcal{O}(-1)|_{Y \setminus X}$. Also let $\mathcal{V}_9 = V_9^*|_{Y \setminus X}$ and $\mathcal{V}_3 = \ker \Phi|_{Y \setminus X}$. Then \mathcal{V}_3 is a rank 3 vector bundle on $Y \setminus X$ satisfying $\mathcal{V}_1 \subset \mathcal{V}_3 \subset \mathcal{V}_9$.

The tangent space at a point $x \in Y \setminus X$ is the kernel of the Jacobian map $\mathcal{V}_9/\mathcal{V}_1 \rightarrow \bigwedge^2(\mathcal{V}_3/\mathcal{V}_1)$ restricted to x (this is similar to the argument in the previous proof). So x is smooth if and only if this map is nonzero. Suppose that the map is zero at x and do a change of basis so that $e_i(x) = 0$ for $i > 1$ and so that $(\mathcal{V}_3)_x$ is defined by $e_i = 0$ for $i > 3$. The entries of the Jacobian matrix are given by the coefficients of $[23i]$ for $i = 4, \dots, 9$, and so those coefficients are 0. This means that the 1-parameter subgroup with weight $(-2, -2, -2, 1, 1, 1, 1, 1, 1)$ destabilizes u , which contradicts that u is stable. So Y is indeed a smooth hypersurface away from X . \square

Recall that given a variety X , its Albanese variety is an abelian variety satisfying a certain universal property (which will not be relevant for our purposes).

Proposition 3.3. *X is a torsor over its Albanese variety J and $\mathcal{O}_X(1)$ is a $(3, 3)$ -polarization.*

Proof. Let R be a DVR whose residue field is \mathbf{k} and whose fraction field K is of characteristic 0. Pick a lift u_R of u to $\bigwedge^3(R^9)$; then u_K is a stable element of $\bigwedge^3(K^9)$ since being non-stable is a closed condition. The construction that we just discussed gives a surface \mathcal{X}_R over R whose generic fiber \mathcal{X}_K is a torsor over its Albanese variety [GSW, Theorem 5.5] and whose special fiber is $\mathcal{X}_{\mathbf{k}} = X$. Let ℓ be a prime different from the characteristic of \mathbf{k} . Then the ℓ -adic Betti numbers of \mathcal{X}_K and X are the same [Mil, Corollary VI.4.2]. We also know that $\omega_X = \mathcal{O}_X$ (from the locally free resolution of \mathcal{O}_X in [GSW, §5.2]). So over \mathbf{k}^{sep} , X is isomorphic to an abelian surface [BM]. In particular, X is a torsor over its Albanese variety (see the proof of [GSW, Theorem 3.1]).

The statement about $\mathcal{O}_X(1)$ is proven in [GSW, Proposition 5.6] for a field of characteristic 0. In particular, after base changing to a finite extension of R , we can find a cube root of $\mathcal{O}_X(1)$ over the generic fiber. This can be extended to a line bundle over the whole family whose cube is $\mathcal{O}_X(1)$ (using properness of the Picard variety), which means that it is a $(3, 3)$ -polarization over the special fiber as well. \square

Since $\mathcal{O}_X(1)$ is a $(3, 3)$ -polarization, the action of $J[3]$ on X extends to an action of $J[3]$ on $\mathbf{P}(V_9^*)$. Let X^i be the Picard variety of line bundles on X whose polarization is of type (i, i) . By [GS2, Theorem 3.6] (although it is stated in characteristic 0, the proof does not rely on this assumption), we have an isomorphism

$$\begin{aligned} X(\mathbf{k}^{\text{sep}}) &\rightarrow X^1(\mathbf{k}^{\text{sep}}) \\ x &\mapsto \mathbf{P}(\ker \Phi(x)) \cap X(\mathbf{k}^{\text{sep}}). \end{aligned}$$

Since Φ is defined over \mathbf{k} , this map descends to an isomorphism $X \rightarrow X^1$ defined over \mathbf{k} . Furthermore, we have a cubing map $X^1 \rightarrow X^3$ and $\mathcal{O}_X(1) \in X^3$ gives us an isomorphism $X^3 \cong J$. Combining this, we have a map

$$\psi: X \rightarrow J$$

which gives X the structure of a 3-covering of J .

The preimage of $\mathcal{O}_X(1)$ under the cubing map $X^1 \rightarrow X^3$ is a torsor for $J[3]$. Each geometric point represents a line bundle \mathcal{L} such that $h^0(X; \mathcal{L}) = 1$, and the zero locus $Z(\mathcal{L})$ of the unique, up to scalar multiple, section is a theta divisor of X . So $Z(\mathcal{L})$ is a genus 2 curve whose Jacobian is X .

Lemma 3.4. *Under the isomorphism $X \rightarrow X^1$, the image of $Z(\mathcal{L})$ contains the point representing \mathcal{L} . Furthermore, this point is a Weierstrass point of $Z(\mathcal{L})$.*

Proof. The first statement is equivalent to $x \in \ker \Phi(x)$. But this follows from the fact that $\Phi(x)$ is the contraction of an alternating trilinear form on V_9 by x .

For the second statement, let P be the point on $Z(\mathcal{L})$. First assume that the characteristic of \mathbf{k} is 0. Then we can check more generally that for any point $x \in X$, we have that x is a Weierstrass point of $\mathbf{P}(\ker \Phi(x)) \cap X$. For this, it suffices to check a single point since the property is invariant under translation, and this is done in [GS2, Remark 3.15].

For the general case, pick a DVR R as in the proof of Proposition 3.3 and a lift u_R of u to $\Lambda^3(R^9)$. Our construction is valid in families, so we get a curve \mathcal{C} over R together with a section $\mathcal{P}: \text{Spec}(R) \rightarrow \mathcal{C}$. Since $\mathcal{M} = \mathcal{O}_{\mathcal{C}}(\mathcal{P})^{\otimes 2}$ extends the canonical bundle on \mathcal{C}_K , we see that $\mathcal{M} = \Omega_{\mathcal{C}/R}^1$. In particular, $\mathcal{M}_{\mathbf{k}} = \omega_{Z(\mathcal{L})}$, and so P is a Weierstrass point. \square

For any two choices $\mathcal{L}, \mathcal{L}'$, $Z(\mathcal{L})$ and $Z(\mathcal{L}')$ differ by translation by an element of $J[3]$, so they have the same image under ψ . So the reduced image of the union of these curves under ψ is a genus 2 curve $C \subset J$ (defined over \mathbf{k}) whose Jacobian is J and $P := \mathcal{O}_X(1) \in C(\mathbf{k})$ is a Weierstrass point.

Using basic properties of finite Heisenberg group schemes, we know that the inclusion $J[3] \subset \mathbf{PGL}(V_9)$ coming from the translation action of $J[3]$ on $\mathbf{P}(V_9^*)$ lifts to an inclusion $J[3] \subset \mathbf{SL}(V_9)/\mu_3$.

Lemma 3.5. *The map $H^1(\mathbf{k}; \mathbf{SL}(V_9)/\mu_3) \rightarrow H^1(\mathbf{k}; \mathbf{PGL}(V_9))$ is injective.*

Proof. We have the following commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_3 & \longrightarrow & \mathbf{SL}_9 & \longrightarrow & \mathbf{SL}_9/\mu_3 \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathbf{GL}_9 & \longrightarrow & \mathbf{PGL}_9 \longrightarrow 1 \end{array}$$

which gives the following commutative diagram

$$\begin{array}{ccc} H^1(\mathbf{k}; \mathbf{SL}_9/\mu_3) & \longrightarrow & H^2(\mathbf{k}; \mu_3) \\ \downarrow & & \downarrow \\ H^1(\mathbf{k}; \mathbf{PGL}_9) & \longrightarrow & H^2(\mathbf{k}; \mathbf{G}_m) \end{array}$$

The horizontal maps are injective since $H^1(\mathbf{k}; \mathbf{SL}_9) = H^1(\mathbf{k}; \mathbf{GL}_9) = 1$ and the right vertical map is injective since $\mathbf{G}_m/\mu_3 \cong \mathbf{G}_m$ and $H^1(\mathbf{k}; \mathbf{G}_m) = 1$. So we conclude that the map $H^1(\mathbf{k}; \mathbf{SL}_9/\mu_3) \rightarrow H^1(\mathbf{k}; \mathbf{PGL}_9)$ is injective. \square

Recall that 3-coverings $\psi: X \rightarrow J$ are classified by cohomology classes $[\psi] \in H^1(\mathbf{k}; J[3])$. To get the cohomology class, note that $\psi^{-1}(0)$ is a torsor under $J[3]$.

Lemma 3.6. $[\psi] \in \ker(H^1(\mathbf{k}; J[3]) \rightarrow H^1(\mathbf{k}; \mathbf{SL}(V_9)/\mu_3))$.

Proof. By Lemma 3.5, it suffices to show that $[\psi]$ is in the kernel of the composition $H^1(\mathbf{k}; J(C)[3]) \rightarrow H^1(\mathbf{k}; \mathbf{PGL}(V_9))$. The map sends the $J[3]$ -torsor $\psi^{-1}(0)$ to the $\mathbf{PGL}(V_9)$ -torsor $\psi^{-1}(0) \times^{J[3]} \mathbf{PGL}(V_9)$. The data of this $\mathbf{PGL}(V_9)$ -torsor is equivalent to the embedding $\psi^{-1}(0) \subset \mathbf{P}(V_9^*)$. Projective space represents the trivial \mathbf{PGL} -torsor, so the image of $[\psi]$ in $H^1(\mathbf{k}; \mathbf{PGL}(V_9))$ is trivial. \square

4. A CONSTRUCTION OF TRIVECTORS

Let C be a smooth genus 2 curve with a marked Weierstrass point $P \in C(\mathbf{k})$. Let $J^1(C)$ be the Picard variety of degree 1 line bundles, and let $J(C)$ be the Jacobian of degree 0 line bundles. We identify $J^1(C) \cong J(C)$ via $\mathcal{L} \mapsto \mathcal{L}(-P)$.

Define $V_9 = H^0(J^1(C); 3\Theta)$. Then $J(C) \subset \mathbf{P}(V_9^*)$ is embedded by a $(3, 3)$ -polarization, denoted $\mathcal{O}(1)$. Define a codimension 1 subvariety of $J(C) \times J(C)$ by

$$X = X_{C,P} = \{(\mathcal{L}_1, \mathcal{L}_2) \mid \text{Hom}_C(\mathcal{L}_1, \mathcal{L}_2(P)) \neq 0\}.$$

The line bundle $\mathcal{O}(1, 1) \otimes \mathcal{O}(-X)$ has divisor class $3\pi_1^*\Theta + 3\pi_2^*\Theta - \Theta_{\text{diag}}$. This is the pullback of a principal polarization on $J(C) \times J(C)$ via the endomorphism

$$\begin{aligned} J(C) \times J(C) &\rightarrow J(C) \times J(C) \\ (a, b) &\mapsto (2a + b, a + 2b). \end{aligned}$$

The kernel of this map is the diagonal copy of $J(C)[3]$ which has degree 81. In particular, $\mathcal{O}(1, 1) \otimes \mathcal{O}(-X)$ has a single cohomology group of dimension $9 = \sqrt{81}$.

Lemma 4.1. $h^0(\mathcal{O}(1, 1) \otimes \mathcal{O}(-X)) = 9$ and all other cohomology groups vanish.

Proof. It suffices to show that $h^0(\mathcal{O}(1, 1) \otimes \mathcal{O}(-X)) \neq 0$. Define a divisor of $J(C) \times J(C)$:

$$D = \{(\mathcal{L}_1, \mathcal{L}_2) \mid h^0(\mathcal{L}_1 \otimes \mathcal{L}_2(-P)) \neq 0 \text{ or } h^0(\mathcal{L}_1^{-1} \otimes \mathcal{L}_2(P)) \neq 0\}.$$

Then D is linearly equivalent to $2\pi_1^*\Theta \otimes 2\pi_2^*\Theta$. In particular, $\mathcal{O}(1, 1) \otimes \mathcal{O}(-D)$ has a nonzero section. But $X \subset D$, so we see that $\mathcal{O}(1, 1) \otimes \mathcal{O}(-X)$ also has a nonzero section. \square

Define

$$W = H^0(J(C) \times J(C); \mathcal{O}(1, 1) \otimes \mathcal{O}(-X)) \subset V_9 \times V_9.$$

By Serre duality and Riemann–Roch, $\text{Hom}_C(\mathcal{L}_1, \mathcal{L}_2(P)) \neq 0$ if and only if $\text{Hom}_C(\mathcal{L}_2, \mathcal{L}_1(P)) \neq 0$, so X is preserved under the involution that swaps the two copies of V_9 .

Let H denote the finite Heisenberg group scheme, i.e., the extension

$$1 \rightarrow \mu_3 \rightarrow H \rightarrow J(C)[3] \rightarrow 1.$$

Then H acts diagonally on $V_9 \otimes V_9$ preserving W . Note that V_9 is the unique irreducible representation of H of weight 1 (see [Se, Appendix]), and W has weight 2, so V_9 and W^* are isomorphic as representations of H . So the inclusion gives an H -equivariant map (well-defined up to scalar multiple) $V_9^* \rightarrow V_9 \otimes V_9$.

Lemma 4.2. *The image of V_9^* is contained in $\bigwedge^2 V_9$.*

Proof. By irreducibility, it suffices to show that a single nonzero element in W is alternating under the involution swapping the two copies of V_9 . Define D as in the proof of Lemma 4.1. Pick a bilinear equation that vanishes on D , i.e., a section of $\mathcal{O}(1, 1) \otimes \mathcal{O}(-D)$. Since the diagonal $J(C)$ is contained in D , if we restrict this equation to the diagonal, we get a section of 4Θ that vanishes on $J(C)$. But we know that such equations are alternating since the Kummer variety has no quadratic polynomials vanishing on it in its 2Θ embedding. \square

So we can represent this map by an element $\gamma = \gamma_{(C,P)} \in V_9 \otimes \bigwedge^2 V_9$.

Lemma 4.3. $\gamma_{(C,P)} \in \bigwedge^3 V_9$.

Proof. Note that γ is an H -invariant element. Furthermore, $\Lambda^2 V_9$ is a weight 2 representation of dimension 36, and hence it is a direct sum of 4 copies of V_9^* [Se, Theorem A.6], so the space of H -invariant vectors in $V_9 \otimes \Lambda^2 V_9$ is 4-dimensional. The space of H -invariant vectors in $\Lambda^3 V_9$ is also 4-dimensional (we can do this calculation in characteristic 0 and then specialize to get ≥ 4 -dimensional), so $\gamma \in \Lambda^3 V_9$. \square

Lemma 4.4. *The projection of $X_{C,P}$ to either copy of \mathbf{P}^8 lies in the rank 4 locus $X(\gamma)$ constructed in §3.*

Proof. Pick a point x in the projection of $X_{C,P}$ to $\mathbf{P}(V_9^*)$. Evaluating γ on x , we get a skew-symmetric matrix $V_9^* \rightarrow V_9$ whose image is the set of linear equations vanishing on the fiber of $X_{C,P}$ over x . But this is a translate of a theta divisor. As an embedded variety, the theta divisor is a genus 2 curve under its tricanonical embedding, and hence satisfies 4 linear equations, so this skew-symmetric map has rank ≤ 4 . \square

Proposition 4.5. *Let $G = \mathbf{SL}(V_9)/\mu_3$.*

- (a) $\gamma_{(C,P)} \in \Lambda^3 V_9$ is stable with respect to the action of G .
- (b) The stabilizer of $[\gamma_{(C,P)}] \in \mathbf{P}(\Lambda^3 V_9)$ in G is isomorphic to $\mathbf{J}(C)[3] \rtimes \text{Aut}(C, P)$ where $\text{Aut}(C, P)$ is the group scheme of \mathbf{k}^{sep} -automorphisms of C which fix P .
- (c) If the characteristic is different from 2 and 5, then the stabilizer of $\gamma_{(C,P)}$ in G is isomorphic to $\mathbf{J}(C)[3]$.
- (d) In characteristic 2, the stabilizer of $\gamma_{(C,P)}$ is isomorphic to $\mathbf{J}(C)[3] \rtimes \mathbf{Z}/2$ if (C, P) is generic, where the $\mathbf{Z}/2$ comes from the hyperelliptic involution on C and acts by the automorphism $g \mapsto g^{-1}$.
- (e) In characteristic 5, the stabilizer of $\gamma_{(C,P)}$ is isomorphic to $\mathbf{J}(C)[3]$ if (C, P) is generic.

Proof. First we calculate the stabilizer $G_{[\gamma]}$ of $[\gamma] \in \mathbf{P}(\Lambda^3 V_9)$. By functoriality, it is clear that $\mathbf{J}(C)[3] \rtimes \text{Aut}(C, P) \subseteq G_{[\gamma]}$. Conversely, let S be a \mathbf{k} -scheme and consider an element $g \in G_{[\gamma]}(S)$. By the construction above, g preserves the embedding of $(\mathbf{J}(C) \times \mathbf{J}(C))(S)$ in $(\mathbf{P}(V_9^*) \times \mathbf{P}(V_9^*))(S)$ and the subvariety $X(S)$. In particular, g acts on $\mathbf{J}(C)(S)$ and preserves the relation $\text{Hom}_{C(S)}(\mathcal{L}_1, \mathcal{L}_2(P)) \neq 0$, which implies that g permutes the elements of $\mathbf{J}(C)[3](S)$. So $G_{[\gamma]}$ is generated by $\mathbf{J}(C)[3]$ and a subgroup of the automorphisms of $\mathbf{J}(C)$ which fixes the identity. Using Torelli's theorem, an automorphism of $\mathbf{J}(C)$ that fixes the identity and the embedding of $\mathbf{J}(C)$ comes from an automorphism of C which fixes P ; since $G_{[\gamma]}$ contains $\mathbf{J}(C)[3] \rtimes \text{Aut}(C, P)$, we deduce that they are equal. This proves (b).

In particular, $G_{[\gamma]}$ is finite. Let $\lambda: G_{[\gamma]} \rightarrow \mathbf{G}_m$ be the eigenvalue associated with the action of $G_{[\gamma]}$ on $[\gamma]$. The stabilizer of γ is $\ker \lambda$. First note that $\mathbf{J}(C)[3] \subseteq \ker \lambda$ since the projective action of $\mathbf{J}(C)[3]$ lifts to a linear action of the Heisenberg group scheme H in $\mathbf{SL}(V_9)$, and we have already explained why H acts trivially on γ .

If the characteristic is different from 3, then over an algebraic closure of \mathbf{k} , we can diagonalize the action of H on V_9 so that γ is contained in a standard Cartan subspace and its normalizer is $\mathbf{Z}/3 \times \mathbf{Sp}_4(\mathbf{Z}/3)$: see [BL, Proposition 6.7.1] for the case of characteristic 0; the structure of $\mathbf{J}(C)[3]$ over an algebraically closed field remains the same outside of characteristic 3, so for the remaining cases we can appeal to [Se, §0]. In particular, $G_{[\gamma]}/(\mathbf{J}(C)[3])$ is a subgroup of $\mathbf{Z}/3 \times \mathbf{Sp}_4(\mathbf{Z}/3)$. Since γ has a finite stabilizer, it does not lie in any reflection hyperplane (Proposition 2.7). By Proposition 2.8, γ is a stable element.

Now we handle characteristic 3. Let \mathcal{M} be the moduli stack (over \mathbf{Z}) of smooth genus 2 curves with a marked Weierstrass point and let \mathcal{X} be the quotient stack $[\mathbf{P}(\Lambda^3 V_9)/\mathbf{PGL}(V_9)]$.

We have constructed a morphism $\gamma: \mathcal{M} \rightarrow \mathcal{X}$ via $C \mapsto \gamma_C$. Let $\mathcal{Z} \subset \mathcal{X}$ be the substack of non-stable elements. By Proposition 2.10, \mathcal{Z} is irreducible and has codimension 1. In fact, its ideal sheaf is locally principal, so the ideal sheaf of $\gamma^{-1}(\mathcal{Z})$ is also locally principal and hence $\gamma^{-1}(\mathcal{Z})$ is either empty or of codimension ≤ 1 . The argument above shows that $\gamma^{-1}(\mathcal{Z}) \times \operatorname{Spec}(\mathbf{Z}[\frac{1}{3}]) = \emptyset$, so $\gamma^{-1}(\mathcal{Z})$ is supported on the prime 3. But $\mathcal{M} \times \operatorname{Spec}(\mathbf{Z}[\frac{1}{3}])$ is irreducible, so either $\gamma^{-1}(\mathcal{Z}) = \emptyset$ or it coincides with $\mathcal{M} \times \operatorname{Spec}(\mathbf{Z}[\frac{1}{3}])$. Taking any smooth curve C_c guarantees that γ_C is stable by Proposition 2.11, so we conclude that $\gamma^{-1}(\mathcal{Z}) = \emptyset$. This proves (a).

Now we prove (c), so we assume that the characteristic is different from 2 and 5. Recall that $G_\gamma = \ker \lambda$, so we need to show that $\operatorname{Aut}(C, P)$ is mapped faithfully via λ . Put (C, P) into Weierstrass normal form

$$(4.5.1) \quad y^2 = x^5 + c_{12}x^3 + c_{18}x^2 + c_{24}x + c_{30}.$$

By degree considerations (where $\deg(y) = 5$ and $\deg(x) = 2$), any automorphism of (C, P) must be of the form

$$\begin{aligned} y &\mapsto a_1^5 y + a_2 x^2 + a_3 x + a_4 \\ x &\mapsto a_1^2 x + a_5 \end{aligned}$$

for some scalars a_1, a_2, a_3, a_4, a_5 and $a_1 \neq 0$. When we do these substitutions to (4.5.1) and subtract (4.5.1), we get a relation on x, y which is of degree < 10 , so which must be identically 0. The coefficients of $x^2 y, xy, y$ on the left side are $2a_1^5 a_2, 2a_1^5 a_3, 2a_1^5 a_4$, respectively, so we conclude that $a_2 = a_3 = a_4 = 0$. Similarly, the coefficient of x^4 on the right side is $5a_1^8 a_5$, so we conclude that $a_5 = 0$.

In particular, the automorphism takes the form

$$y \mapsto a_1^5 y, \quad x \mapsto a_1^2 x$$

for some ℓ th root of unity a_1 (since the automorphism has finite order). Again, do the substitution to (4.5.1), divide by a_1^{10} and subtract (4.5.1). Then we get

$$c_{12}(a_1^{-4} - 1)x^3 + c_{18}(a_1^{-6} - 1)x^2 + c_{24}(a_1^{-8} - 1)x + c_{30}(a_1^{-10} - 1) = 0,$$

so the left hand side must be identically 0. If $\ell \notin \{1, 2, 4, 5, 8, 10\}$, then $c_{12} = c_{24} = c_{30} = 0$. But then (4.5.1) is $y^2 = x^2(x^3 + c_{18})$, which is a singular curve. So we only need to show that λ maps $\mu_\ell \subset \operatorname{Aut}(C, P)$ faithfully where $\ell \in \{1, 2, 4, 5, 8, 10\}$; it suffices to consider the cases $\ell = 2$ and $\ell = 5$.

For $\ell \in \{2, 5\}$, let \mathcal{M}_ℓ be the space of curves with an action of μ_ℓ as described above. Then \mathcal{M}_ℓ is an irreducible stack over $\mathbf{Z}[1/\ell]$. Indeed, the action of μ_ℓ must survive completing the ℓ -th power in the curve, and this forces the action to be diagonal in the variables. Thus \mathcal{M}_5 is the irreducible stack of curves of the form $x^2 + z^5 + c_{15}x + c_{30} = 0$ modulo $x \mapsto x + a$ (with μ_5 acting by $z \mapsto \zeta_5 z$) and \mathcal{M}_2 is the irreducible stack of curves of the form

$$x^2 + z^5 + c_6 z^4 + c_{12} z^3 + c_{18} z^2 + c_{24} z + c_{30},$$

modulo $z \mapsto z + a$ (with μ_2 acting by $x \mapsto -x$).

Let \mathcal{C} be the universal curve over \mathcal{M}_ℓ . By composing λ with the natural morphism $\mu_\ell \rightarrow \operatorname{Aut}(\mathcal{C})$, we obtain a scheme morphism from \mathcal{M}_ℓ to the dual group $\mu_\ell^\vee \cong \mathbf{Z}/\ell$. Since \mathcal{M}_ℓ is irreducible, this morphism must be constant, and thus may be computed in characteristic 0. In this case, any point in the Cartan subspace which is not in the union of the reflection

hyperplanes has a trivial stabilizer. In particular, the stabilizer of γ is isomorphic to $J(C)[3]$. So faithfulness of λ in characteristic 0 implies faithfulness of the restriction to μ_ℓ over $\mathbf{Z}[1/\ell]$.

If (C, P) is generic, then $\text{Aut}(C, P) \cong \mathbf{Z}/2$ and is generated by the hyperelliptic involution ι_C (via Torelli's theorem, this is equivalent to the statement that the generic principally polarized Jacobian has automorphism group $\mathbf{Z}/2$, which is [KS, Lemma 11.2.6]). The induced action of ι_C on $J(C)[3]$ is the inverse map and, if \mathbf{k} has characteristic 0, we can calculate explicitly in a standard Cartan (see, for example, [GS2, (3.2)]) that $\lambda(\iota_C) = -1$, so $\iota_C \notin \ker \lambda$. By semicontinuity, the same is true in any characteristic different from 2. In characteristic 2, the restriction of λ to ι_C is trivial since its image is in $\mu_2 \subset \mathbf{G}_m$, which is non-reduced, while ι_C generates a subgroup isomorphic to $\mathbf{Z}/2$. So $\iota_C \in \ker \lambda$. This proves (d) and (e). \square

5. PUTTING IT ALL TOGETHER

Let $G = \mathbf{SL}(V_9)/\mu_3$ and let G_γ be the stabilizer subgroup of $\gamma \in \Lambda^3 V_9$.

Proposition 5.1. *Pick (C, P) and (C', P') so that we have elements $\gamma = \gamma_{(C,P)} \in \Lambda^3 V_9$ and $\gamma' = \gamma_{(C',P')} \in \Lambda^3 V'_9$. Suppose that there is a linear isomorphism $\varphi: V_9 \cong V'_9$ that sends the line generated by $\gamma_{(C,P)}$ to the line generated by $\gamma_{(C',P')}$. Then there exists an isomorphism $(C, P) \cong (C', P')$.*

Proof. Using φ , we can embed $X_{C,P}$ and $X_{C',P'}$ in the same $\mathbf{P}^8 \times \mathbf{P}^8$, in such a way that their images satisfy the same 9 bilinear equations $W_\gamma = W_{\gamma'}$. Now, consider the projection onto the first \mathbf{P}^8 . By Lemma 4.4, the image of $X_{C,P}$ in \mathbf{P}^8 maps into the rank 4 locus $X(\gamma)$, which is a torsor over an abelian surface (Proposition 3.3). By Hilbert polynomial considerations the image is equal to $X(\gamma)$. The same applies to $X_{C',P'}$, so in particular, we find that φ defines an isomorphism $J^1(C) \cong J^1(C')$ which identifies the respective 3Θ line bundles. Moreover, since the 4 linear equations vanishing on a translate of a theta divisor actually cut out the theta divisor, we find that φ also identifies $X_{C,P}$ and $X_{C',P'}$. Finally, we can recover C as the fiber of $X_{C,P} \rightarrow J(C)$ over 0, and P as the point $(0, 0)$, and similarly for (C', P') . \square

Proposition 5.2. *If we apply the construction of §3 to $\gamma_{(C,P)}$, then the torsor $X(\gamma_{(C,P)})$ is trivial, and (C, P) is the marked curve that comes from the construction in §3.*

Proof. This was shown in the proof of Proposition 5.1. \square

Lemma 5.3. *Let $\gamma \in \Lambda^3 V_9$ be a stable element. Then γ can be recovered from the 9-dimensional space of bilinear forms $W_\gamma \subset \Lambda^2 V_9$ up to scalar multiple.*

Proof. We represent this space as an injective map $W_\gamma \rightarrow \Lambda^2 V_9$. Since γ is stable, $Y(\gamma)$ is a cubic hypersurface, and so the generic element in W_γ has rank 8, and hence its kernel is a line in V_9^* . This gives a rational map $\mathbf{P}(W_\gamma) \dashrightarrow \mathbf{P}(V_9^*)$. Pick 10 elements in W_γ with rank 8 such that any 9 of them are linearly independent. Using these 10 elements, we can lift the rational map to a linear isomorphism $W_\gamma \rightarrow V_9^*$ which is well-defined up to scalar multiple. In particular, we get an element $\gamma': V_9^* \rightarrow \Lambda^2 V_9$ which satisfies $v \in \ker \gamma'(v)$, and so comes from an element in $\Lambda^3 V_9$.

In particular, for each v , either $\gamma'(v)$ has rank ≤ 6 , or it is the unique, up to multiple, skew-symmetric matrix whose kernel is the line spanned by v . Note that $\gamma: V_9^* \rightarrow \Lambda^2 V_9$ has the same property and both images are the same. Pick a basis v_1, \dots, v_9 for V_9^* such that $\gamma(v_i)$ has rank 8 for $i = 1, \dots, 9$. Then there are nonzero scalars α_i so that $\alpha_i \gamma'(v_i) =$

$\gamma(v_i)$. Suppose that they are not all the same scalar, for example, $\alpha_1 \neq \alpha_2$. Note that $\gamma(xv_1 + yv_2) = \gamma'(x\alpha_1 v_1 + y\alpha_2 v_2)$ for all $x, y \in \mathbf{k}$. In particular, $\gamma(xv_1 + yv_2)$ has rank 8 when $x = 0$ or $y = 0$ and has rank ≤ 6 otherwise: the kernel of $\gamma(xv_1 + yv_2)$ contains $xv_1 + yv_2$ and $x\alpha_1 v_1 + y\alpha_2 v_2$, and they are linearly independent when $x, y \neq 0$. But the rank of a family of matrices takes its maximal value on an open set, so this is not possible, and we conclude that all α_i are the same. \square

Proposition 5.4. *Pick stable elements $\gamma, \gamma' \in \bigwedge^3 V_9$ with trivial cohomology class, i.e., $[\psi] = [\psi'] = 0$. Let (C, P) and (C', P') be the marked curves constructed in §3 and assume there is an isomorphism $(C, P) \cong (C', P')$ defined over \mathbf{k} . Then the lines spanned by γ and γ' are in the same $\mathbf{PGL}(V_9)$ -orbit.*

Proof. We have torsors $X(\gamma), X(\gamma') \subset \mathbf{P}(V_9^*)$. Since their cohomology classes are trivial, we can find an embedding $C \subset X(\gamma)$ and $C' \subset X(\gamma')$ so that their Weierstrass points lie on the 3-torsion (more specifically, the construction in §3 shows that if a 3-torsion point is \mathbf{k} -rational, then there is a \mathbf{k} -rational theta divisor containing it). In particular, the isomorphism $(C, P) \cong (C', P')$ induces an isomorphism $X(\gamma) \cong X(\gamma')$ that preserves 3Θ , so up to a linear change of coordinates for one of the embeddings, we have $X(\gamma) = X(\gamma')$. In particular, there is an identification of their Poincaré divisors, which then satisfy the same 9 bilinear equations, i.e., $W_\gamma = W_{\gamma'}$. Lemma 5.3 implies that γ and γ' are equal up to scalar multiple after the change of coordinates. \square

Theorem 5.5. *The construction in §3 is a bijection between the stable orbits of $\mathbf{P}(\bigwedge^3 V_9)$ under the action of $\mathbf{PGL}(V_9)$ and the set of \mathbf{k} -isomorphism classes of triples (C, P, ψ) where C is a smooth genus 2 curve, $P \in C(\mathbf{k})$ is a Weierstrass point, and $\psi \in \ker(H^1(\mathbf{k}; J(C)[3]) \rightarrow H^1(\mathbf{k}; \mathbf{PGL}(V_9)))$.*

Proof. Given a smooth genus 2 curve with Weierstrass point P , we have constructed a stable element in $\mathbf{P}(\bigwedge^3 H^0(J(C); 3\Theta)^*)$ in §4. If we pick a linear isomorphism $H^0(J(C); 3\Theta)^* \cong V_9$, we hence get an element of $\mathbf{P}(\bigwedge^3 V_9)$. The $\mathbf{PGL}(V_9)$ -orbit of this element does not depend on the choice of isomorphism. So we have a well-defined map Φ from the set of \mathbf{k} -isomorphism classes of (C, P) to $\mathbf{PGL}(V_9)$ -orbits in $\mathbf{P}(\bigwedge^3 V_9)$. Furthermore, by Proposition 5.2, $\Phi(C, P)$ has trivial cohomology class. By Proposition 5.1, this map is injective on \mathbf{k} -isomorphism classes of (C, P) .

In §3, we constructed a map from $\mathbf{PGL}(V_9)$ -orbits of $\mathbf{P}((\bigwedge^3 V_9)_{\text{st}})$ to the set of \mathbf{k} -isomorphism classes of (C, P) ; let Ψ be the restriction to the orbits with trivial cohomology class. By Proposition 5.2, $\Psi \circ \Phi$ is the identity, so Ψ is surjective. By Proposition 5.4, Ψ is injective, so Φ is a bijection between \mathbf{k} -isomorphism classes of marked curves (C, P) and $\mathbf{PGL}(V_9)$ -orbits of stable elements in $\mathbf{P}(\bigwedge^3 V_9)$ with trivial cohomology class.

By Proposition 4.5, the stabilizer of any element in $\Phi(C, P)$ is isomorphic to $J(C)[3] \rtimes \text{Aut}(C, P)$. In particular,

$$\ker(H^1(\mathbf{k}; J(C)[3] \rtimes \text{Aut}(C, P)) \rightarrow H^1(\mathbf{k}; \mathbf{PGL}(V_9)))$$

is in bijection with the $\mathbf{PGL}(V_9)$ -orbits in $\mathbf{P}(\bigwedge^3 V_9)$ which are in the same orbit as $\Phi(C, P)$ over a separable closure of \mathbf{k} . Now consider the map

$$H^1(\mathbf{k}; J(C)[3] \rtimes \text{Aut}(C, P)) \rightarrow H^1(\mathbf{k}; \text{Aut}(C, P)).$$

The latter group parametrizes \mathbf{k} -forms of C , so each such orbit is naturally associated to a \mathbf{k} -form of C . In particular, the orbits that correspond to C itself, i.e., \mathbf{k} -forms that are

actually isomorphic to C over \mathbf{k} , are in bijection with the kernel of

$$\ker(H^1(\mathbf{k}; J(C)[3]) \rightarrow H^1(\mathbf{k}; \mathbf{PGL}(V_9))).$$

In particular, Φ extends to a map on triples (C, P, ψ) and gives an isomorphism to all stable $\mathbf{PGL}(V_9)$ -orbits in $\mathbf{P}(\Lambda^3 V_9)$. \square

Corollary 5.6. *If \mathbf{k} is algebraically closed of characteristic different from 3, then every stable element of $\Lambda^3 V_9$ is in the standard Cartan subspace up to the action of G .*

Proof. By the construction in §4, every stable element of the form $\gamma_{(C,P)}$ is in the standard Cartan subspace up to the action of G . By Theorem 5.5, they all arise in this way. \square

6. COMPLEMENTS

6.1. Selmer groups. For this section, suppose that \mathbf{k} is a global field, and let B be an abelian variety defined over \mathbf{k} . Let $\alpha \in H^1(\mathbf{k}; B[n])$ be a torsor for $B[n]$. We can use this to twist the multiplication by n map $B \xrightarrow{n} B$ to get $B' \rightarrow B$ where B' is a B -torsor. We say that α is an element of the n -**Selmer group** of B if, for all completions \mathbf{k}_v of \mathbf{k} , the corresponding torsor B' has a \mathbf{k}_v -rational point. We denote this subgroup by $\text{Sel}_n(B) \subset H^1(\mathbf{k}; B[n])$.

Proposition 6.1. *Let C be a genus 2 curve with rational Weierstrass point. The 3-Selmer group $\text{Sel}_3(J(C))$ is contained in $\ker(H^1(\mathbf{k}; J(C)[3]) \rightarrow H^1(\mathbf{k}; \mathbf{SL}_9/\mu_3))$.*

Proof. Pick $\psi \in \text{Sel}_3(J(C))$. Then ψ gives an embedding $X \subset S$ where S is a Brauer–Severi variety of dimension 8 and X is the corresponding twist of $J(C)$. By assumption, $X(\mathbf{k}_v) \neq \emptyset$ for all completions \mathbf{k}_v of \mathbf{k} . Brauer–Severi varieties satisfy the Hasse principle, so we conclude that $S \cong \mathbf{P}^8$ and that the image of ψ in $H^1(\mathbf{k}; \mathbf{PGL}(9))$ is trivial. By Lemma 3.5, its image in $H^1(\mathbf{k}; \mathbf{SL}(9)/\mu_3)$ is also trivial. \square

In particular, triples (C, P, ψ) where C is a genus 2 curve, $P \in C(\mathbf{k})$ is a Weierstrass point, and $\psi \in \text{Sel}_3(J(C))$ are parametrized by certain $\mathbf{PGL}(V_9)$ -orbits in $\mathbf{P}(\Lambda^3 V_9)$.

6.2. Ordinary curves. Let \mathbf{k} be a field of characteristic 3. Given a smooth curve of genus g , then $|J(C)(\mathbf{k}^{\text{sep}})| = 3^r$ where $0 \leq r \leq g$. If $r = g$, then C is **ordinary**.

The Lie algebra of type E_8 has a cubing map $x \mapsto x^{[3]}$ which induces a cubing map $\Lambda^3 V_9 \rightarrow \mathfrak{sl}(V_9)$.

Set γ_0 to be the principal nilpotent element with all $c_i = 0$ Proposition 2.1:

$$\gamma_0 = [267] + [258] + [348] + [169] + [357] + [249] + [178] + [456].$$

Lemma 6.2. *Let C be the genus 2 curve associated with a stable element $\gamma \in \Lambda^3 V_9$. The Lie algebra of the stabilizer of γ (equivalently, the Lie algebra of $J(C)[3]$) is 2-dimensional, and is spanned by $\gamma^{[3]}$ and $\gamma^{[9]}$.*

Proof. By a direct calculation [Sp], we have $\dim(\ker \text{ad } \gamma_0 \cap \mathfrak{g}_0) = 2$; one can also verify by an explicit calculation that the centralizer is spanned by $\gamma_0^{[3]}$ and $\gamma_0^{[9]}$. By semicontinuity, the Lie algebra of $J(C)[3]$ coming from γ is at most 2-dimensional. However, a generic γ (for example, take a stable element in the Cartan subspace) comes from an ordinary curve, in which case the Lie algebra is 2-dimensional, so the dimension is always 2, and agrees with the ≥ 2 -dimensional span of $\gamma^{[3]}$ and $\gamma^{[9]}$. \square

Corollary 6.3. *Pick a stable element $\gamma \in \Lambda^3 V_9$. The 3-rank of the associated curve is:*

- (a) 2 if $\gamma^{[3]}$ is semisimple,
- (b) 1 if $\gamma^{[3]}$ is not semisimple, but $\gamma^{[9]}$ is semisimple,
- (c) 0 if neither $\gamma^{[3]}$ nor $\gamma^{[9]}$ is semisimple.

Proof. The Weil pairing shows that $J(C)[3]^\vee \cong J(C)[3]$. In particular, the 3-rank r appears in the reduced quotient $(\mathbf{Z}/3)^r$ of $J(C)[3]$ and hence appears in the largest diagonalizable subgroup $\mu_3^r \subset J(C)[3]$. So the 3-rank of the curve C is the dimension of the largest semisimple subalgebra of the Lie algebra of $J(C)[3]$. \square

Remark 6.4. We can write our curve C in Weierstrass normal form

$$y^2 = x^5 + c_{12}x^3 + c_{18}x^2 + c_{24}x + c_{30}.$$

According to [EP, Lemma 2.2], the 3-rank of C is:

$$\begin{cases} 2 & \text{if } c_{24} \neq 0, \\ 1 & \text{if } c_{24} = 0, \ c_{18} \neq 0, \\ 0 & \text{if } c_{24} = c_{18} = 0. \end{cases}$$

Furthermore, by Lemma 6.2, we know that $\gamma^{[27]}$ is a linear combination of $\gamma^{[3]}$ and $\gamma^{[9]}$; in Weierstrass normal form, a computer calculation shows that

$$\gamma^{[27]} = c_{24}\gamma^{[3]} - c_{18}\gamma^{[9]}.$$

\square

6.3. Model for 3-torsion. Let $\gamma \in \bigwedge^3 V_9$ be a stable vector. By Proposition 4.5, the stabilizer of $[\gamma] \in \mathbf{P}(\bigwedge^3 V_9)$ in $\mathbf{SL}(V_9)/\mu_3$ is isomorphic to $J(C)[3] \rtimes \text{Aut}(C, P)$ where (C, P) is the marked curve associated to γ , and there is also an associated torsor of $J(C)[3]$. Here is a more direct construction for this torsor.

The split Lie algebra of type E_8 has a graded direct sum decomposition

$$\mathfrak{sl}(V_9) \oplus \bigwedge^3 V_9 \oplus \bigwedge^6 V_9.$$

Pick a flag of subspaces $F_1 \subset F_3 \subset F_6 \subset F_8 \subset V_9$ (the subscripts indicate the dimension of the subspace). Via the embedding $\text{Flag}(1, 8; V_9) \subset \mathbf{P}(\mathfrak{sl}(V_9))$, the subspaces $F_1 \subset F_8$ determine (up to scalar multiple) an element $v_0 \in \mathfrak{sl}(V_9)$, F_3 determines an element $v_1 \in \bigwedge^3 V_9$, and F_6 determines an element $v_2 \in \bigwedge^6 V_9$. We say that F_\bullet is **compatible with** γ if:

- (a) $[v_0, \gamma] \in \bigwedge^3 V_9$ is a scalar multiple of v_1 ,
- (b) $[v_1, \gamma] \in \bigwedge^6 V_9$ is a scalar multiple of v_2 , and
- (c) $[v_2, \gamma] \in \mathfrak{sl}(V_9)$ contains F_6 in its kernel and its image is contained in F_1 .

The conditions above are algebraic, so determines a subscheme $F(\gamma)$ of compatible flags. To be precise, let F_\bullet be the standard flag defined by $F_i = \langle e_1, \dots, e_i \rangle$. If it is compatible with γ , then it implies that the coefficient of $e_i \wedge e_j \wedge e_k$ vanishes where ijk is

$$(6.5) \quad \begin{aligned} &ij9, & 4 \leq i < j \leq 8; \\ &ij9, & i = 2, 3; \quad 4 \leq j \leq 8; \\ &i78, & 2 \leq i \leq 6; \\ &ij7, ij8, & 4 \leq i < j \leq 6. \end{aligned}$$

Let P be the stabilizer in $\mathbf{GL}(V_9)$ of the standard flag $F_1 \subset F_3 \subset F_6 \subset F_8$. The span of the monomials which are not listed above forms a P -submodule of $\bigwedge^3 V_9$, and via algebraic

induction from P to $\mathbf{GL}(V_9)$, we get a subbundle $\xi \subset \Lambda^3 V_9 \times \text{Flag}(1, 3, 6, 8; V_9)$. Hence, γ is a section of the quotient bundle η , which is of rank 31, and $F(\gamma)$ is the zero locus of this section, and this can be used to define it as a scheme.

Define a $\mathbf{GL}(V_9)$ -equivariant map $\pi: \text{Flag}(1, 3, 6, 8; V_9) \rightarrow \mathbf{P}(V_9^*)$ by sending F_\bullet to the annihilator of F_8 in V_9^* .

Lemma 6.6. *$\pi(F(\gamma))$ is the underlying set of the torsor for $J(C)[3]$ constructed previously. In fact, π gives a bijection between the underlying sets.*

Proof. Fix a compatible flag F_\bullet . Pick nonzero $u \in F_1$ and pick nonzero $x \in V_9^*$ which annihilates F_8 . The action of v_0 on γ can be obtained by first contracting γ by x and then multiplying by u . By assumption, the result is a pure trivector, say equal to $u \wedge \ell_1 \wedge \ell_2$ for $\ell_1, \ell_2 \in V_9$. Let $\Phi(x)$ be the contraction of γ by x . Then $\Phi(x) = \ell_1 \wedge \ell_2 + \ell_3 \wedge u$ for some $\ell_3 \in V_9$ and so $x \in X(\gamma)$. So $\ker \Phi(x) \cap X$ gives a divisor D on X by [GS2, Theorem 3.6], and we want to show that $3D$ is the divisor corresponding to $\mathcal{O}_X(1)$.

To do this, it suffices to show that there is a hyperplane $H \subset \mathbf{P}(V_9^*)$ such that $H \cap X = \ker \Phi(x) \cap X$ as sets. We claim that this works if H is the zero locus of u . It follows from the definition that $\ker \Phi(x) \cap X \subseteq H \cap X$. The correctness of this statement is unaffected if we do a change of basis and if we pass to an algebraic closure of \mathbf{k} . So we do both and assume that F_\bullet is the standard flag. This implies that $\Phi(x) = e_1 \wedge \ell + e_2 \wedge e_3$ where ℓ is in the span of e_2, \dots, e_8 but is not contained in the span of e_2 and e_3 . In particular, doing a further change of basis using the stabilizer of F_\bullet , we may assume that $\ell = e_6$ or $\ell = e_8$. In both cases, we can verify, for generic γ , using a computer algebra system, that if $y \in H \cap X$, then $y \in \ker \Phi(x)$. The general case follows because $H \cap X$ contains C so cannot possibly degenerate any further unless it increases in dimension (but X is not contained in a hyperplane).

For the last statement, let x be a \mathbf{k}^{sep} -point in the image of π . From the proof above, we see that x determines the subspace F_8 in the flag. Also, there is a hyperplane $H \subset \mathbf{P}(V_9^*)$ such that $H \cap X = \ker \Phi(x) \cap X$ as sets. Since X is not contained in a hyperplane, H is unique with this property, and it determines F_1 . If F_\bullet is a compatible flag, then F_3 is determined by $F_1 \subset F_8$ and F_6 is determined by F_3 , so we are done. \square

Theorem 6.7. *$F(\gamma)$ is a degree 81 scheme of dimension 0. In particular, π restricts to a $J(C)[3]$ -equivariant isomorphism between $F(\gamma)$ and the torsor for $J(C)[3]$, so $F(\gamma)$ is reduced outside of characteristic 3.*

Proof. Note that $\dim \text{Flag}(1, 3, 6, 8; V_9) = 31 = \text{rank}(\eta)$, and Lemma 6.6 shows that $F(\gamma)$ is 0-dimensional whenever γ is stable. Hence the degree of $F(\gamma)$ can be calculated as the top Chern class of η , which can be shown to be 81 as follows. The Borel presentation for the (rational) Chow ring of $\text{Flag}(1, 3, 6, 8; V_9)$ describes it as the subring of $S_1 \times S_2 \times S_3 \times S_2 \times S_1$ -invariants inside the quotient ring $\mathbf{Q}[x_1, \dots, x_9]/I$ where I is generated by all positive degree homogeneous S_9 -invariants (S_9 is the symmetric group on 9 letters, and acts by permuting the x_i ; $S_1 \times S_2 \times S_3 \times S_2 \times S_1$ is the subgroup where the first S_2 permutes x_2, x_3 , S_3 permutes x_4, x_5, x_6 , and the second S_2 permutes x_7, x_8). Over the full flag variety of V_9 , the bundle η is filtered by line bundles, one for each monomial in (6.5) and the (rational) Chow ring of the full flag variety is $\mathbf{Q}[x_1, \dots, x_9]/I$. In the Borel presentation, the Chern class of the line bundle corresponding to the monomial ijk is represented by $x_i + x_j + x_k$. So the top Chern class of η is the product of these linear forms, which is $81m$ modulo I where m is a nonzero monomial of degree 31 (doing this in Macaulay2 [GS1], we get $81x_2x_3x_4^3x_5^3x_6^3x_7^6x_8^6x_9^8$).

The $J(C)[3]$ -equivariance of π comes from the fact that $J(C)[3]$ is a subgroup of the stabilizer of γ . \square

In particular, in every \mathbf{k} -orbit in $\bigwedge^3 V_9$, there is a distinguished orbit corresponding to elements which have a compatible flag defined over \mathbf{k} .

Corollary 6.8. *In characteristics different from 2 and 5, $\mathbf{SL}(V_9)/\mu_3$ acts freely on the scheme of pairs (γ, F_\bullet) where $\gamma \in \bigwedge^3 V_9$ is stable and F_\bullet is a compatible flag for γ .*

Remark 6.9. If we permute the basis via 974852631, then the family in Proposition 2.1 becomes

$$\begin{aligned} & [348] - [357] + [267] - [189] + [456] + [239] - [147] - [258] \\ & + c_3[345] - c_6[234] + c_9[127] - c_{12}[124] - c_{15}[356] - c_{18}[236] + c_{24}[126] - c_{30}[136] \end{aligned}$$

and the standard coordinate flag is compatible with the entire family. \square

REFERENCES

- [BG] Manjul Bhargava, Benedict H. Gross, Arithmetic invariant theory, *Symmetry: representation theory and its applications*, 33–54, Progr. Math. **257**, Birkhäuser/Springer, New York, 2014.
- [BL] Christina Birkenhake, Herbert Lange, *Complex Abelian Varieties*, second edition, Grundlehren der Mathematischen Wissenschaften **302**, Springer-Verlag, Berlin, 2004.
- [BM] E. Bombieri, D. Mumford, Enriques’ classification of surfaces in char. p . II, *Complex analysis and algebraic geometry*, pp. 23–42, Iwanami Shoten, Tokyo, 1977.
- [EV] A. G. Èlashvili, E. B. Vinberg, Classification of trivectors of a 9-dimensional space, *Trudy Sem. Vektor. Tenzor. Anal.* **18** (1978), 197–233.
- [El] Noam D. Elkies, The identification of three moduli spaces, [arXiv:math/9905195v1](https://arxiv.org/abs/math/9905195v1).
- [EP] Arsen Elkin, Rachel Pries, Hyperelliptic curves with a -number 1 in small characteristic, *Albanian J. Math.* **1** (2007), no. 4, 245–252.
- [GGR] Skip Garibaldi, Robert Guralnick, Eric Rains, work in preparation.
- [GS1] Daniel R. Grayson, Michael E. Stillman, Macaulay 2, a software system for research in algebraic geometry, version 1.4. Available at <http://www.math.uiuc.edu/Macaulay2/>.
- [GS2] Laurent Gruson, Steven V Sam, Alternating trilinear forms on a 9-dimensional space and degenerations of $(3, 3)$ -polarized Abelian surfaces, *Proc. London Math. Soc. (3)* **110** (2015), no. 3, 755–785, [arXiv:1301.5276v2](https://arxiv.org/abs/1301.5276v2).
- [GSW] Laurent Gruson, Steven V Sam, Jerzy Weyman, Moduli of Abelian varieties, Vinberg θ -groups, and free resolutions, *Commutative Algebra* (edited by Irena Peeva), 419–469, Springer, 2013, [arXiv:1203.2575v2](https://arxiv.org/abs/1203.2575v2).
- [KS] Nicholas M. Katz, Peter Sarnak, *Random matrices, Frobenius eigenvalues, and monodromy*, American Mathematical Society Colloquium Publications **45**, American Mathematical Society, Providence, RI, 1999.
- [Mil] James S. Milne, *Étale Cohomology*, Princeton Mathematical Series **33**, Princeton University Press, 1980.
- [Min] Nguyễn Quang Minh, Vector bundles, dualities and classical geometry on a curve of genus two, *Internat. J. Math.* **18** (2007), no. 5, 535–558, [arXiv:math/0702724v1](https://arxiv.org/abs/math/0702724v1).
- [O] Angela Ortega, On the moduli space of rank 3 vector bundles on a genus 2 curve and the Coble cubic, *J. Algebraic Geom.* **14** (2005), no. 2, 327–356.
- [RS] Eric M. Rains, Steven V Sam, Vector bundles on genus 2 curves and trivectors, [arXiv:1605.04459v1](https://arxiv.org/abs/1605.04459v1).
- [Se] Tsutomu Sekiguchi, On projective normality of abelian varieties II, *J. Math. Soc. Japan* **29** (1977), no. 4, 709–727.
- [Sk] Alexei Skorobogatov, *Torsors and Rational Points*, Cambridge Tracts in Mathematics **144**, Cambridge University Press, Cambridge, 2001.
- [Sp] T. A. Springer, Some arithmetical results on semi-simple Lie algebras, *Publ. Math. IHES* **30** (1966), 115–142.

- [V] È. B. Vinberg, The Weyl group of a graded Lie algebra, *Math. USSR-Izv.* **10** (1976), no. 3, 463–495.
- [W] Jerzy Weyman, *Cohomology of Vector Bundles and Syzygies*, Cambridge University Press, Cambridge, 2003.

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